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On the Compatibility between a Graph and a Simple Order

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Following a definition of Goodman [2], we define a notion of compatibility between an (unoriented) graph and a simple (linear) order. The notion of indifference graph, introduced in [6] for finite graphs, is extended to graphs of arbitrary cardinalities; and the graphs compatible with some simple order are characterized as precisely the indifference graphs. The uniqueness of the compatible simple order is investigated, and it is shown that there is "essentially" only one such for each indifference graph. A definition of compatibility between oriented graphs and simple orders is also introduced and the oriented graphs compatible with some simple order are characterized as the semiorders of Luce [5] and Scott and Suppes [7]. It is proved that there is essentially only one simple order compatible with each semiorder. Finally, the compatibility results are applied to solve the psychologically-motivated problem of representing a graph (oriented graph) by "just noticeable difference" intervals on the real line. In the work on infinite graphs, the Axiom of Choice is freely (and tacitly) assumed.

1. INTRODUCTION

Suppose A is a subset of the set of real numbers and I is the reflexive, symmetric binary relation on A defined by $xIy \leftrightarrow |x - y| \leqslant 1$. More-

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over, suppose R denotes the simple[1] (linear) order "$\leqslant$" on A. Then I and R are *compatible* in the sense that for all $x, y, z \in A$,

$$xRyRz \;\&\; xIz \rightarrow xIy \;\&\; yIz. \tag{1}$$

We are interested in studying the relationship (1) in the abstract.

Suppose now that A is an abstract set of *points* and I a reflexive, symmetric binary relation of *adjacency* on A. We shall call such a pair (A, I) an (unoriented) *graph*,[2] making no restriction on the cardinality of the set A.[3] The problem we consider in the following is this: under what circumstances is there a simple order R on A compatible with I in the sense that equation (1) is satisfied for all x, y, z in A? (Figure 1 shows

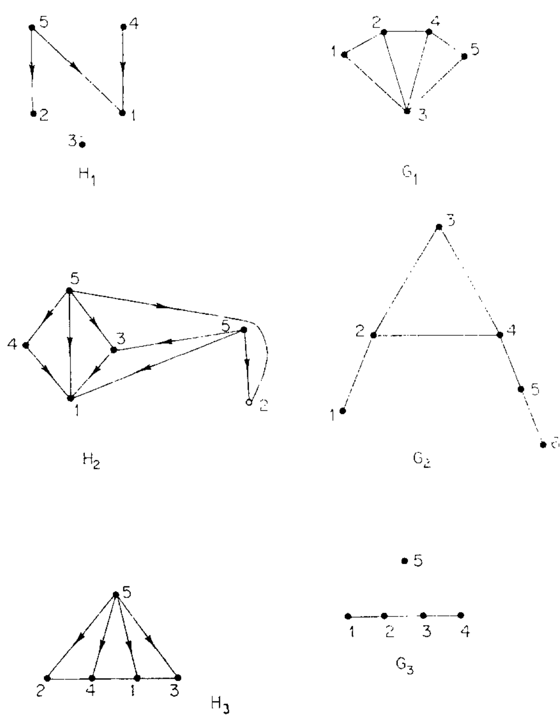

FIGURE 1

[1] (A, R) is a simple order if it is reflexive, transitive, antisymmetric, and complete in the sense that for all x, y in A, xRy or yRx.

[2] Reflexivity of I implies that our graphs have a loop at each point. This assumption is purely a matter of convenience.

[3] One of the features of this paper is that the main results hold without modification for infinite graphs.

three graphs G_1, G_2, G_3 (with loops omitted) and compatible simple orders for each, namely, 1, 2, 3, 4, 5, etc.)

In [2], Goodman asks, given (A, I), whether there is a simple order R on A so that for all $x, y, u, v \in A$,

$$xRuRvRy \;\&\; xIy \to uIv. \tag{2}$$

It is easy to see that this is equivalent to our question. We begin in the next section by describing the class of graphs compatible with some simple order. In Section 3 we prove that the compatible simple order is essentially unique. Then we prove similar theorems for oriented graphs. Finally, these compatibility results will be applied to a psychologically motivated representation problem for graphs (oriented graphs).

2. Characterization of Graphs Having Compatible Simple Orders

We repeat here for convenience several definitions introduced in [6]. We define an equivalence relation E on the points of a graph $G = (A, I)$ by $xEy \leftrightarrow (\forall z)(xIz \leftrightarrow yIz)$. Note that, since our graphs are reflexive, equivalent points are adjacent. Unless otherwise obvious from context, "equivalent" will always mean under E. $[x]$ will denote the equivalence class containing the point x. $G^* = (A^*, I^*)$ will denote the graph obtained by cancelling out the equivalence relation E, i.e., the points are the equivalence classes and adjacency holds between equivalence classes if and only if it holds between their representatives. Finally, if $G \cong G^*$, we shall say G is *reduced.*

In [6] we characterized finite graphs (A, I) for which there is a real-valued function f on A so that for all $x, y \in A$,

$$xIy \leftrightarrow |f(x) - f(y)| \leqslant 1. \tag{3}$$

If the function f were one-to-one, then the graph (A, I) would be compatible with the simple order R on A defined by $xRy \leftrightarrow f(x) \leqslant f(y)$. In general, R so defined is not a simple but a *weak order*, i.e., it is reflexive, transitive, and complete in the sense that for all x, y in A, xRy or yRx. It is sometimes convenient to study compatibility as defined by Eq. (1) between weak orders and graphs as well as between simple orders and graphs. It turns out, however, that every graph compatible with a weak order is also compatible with a simple order. For if (A, I) is compatible with the weak order R, then the relation R^* on A^* defined by

$$[x]\, R^*[y] \leftrightarrow (xRy \vee [x] = [y]) \tag{4}$$

is well defined and a simple order on A^* compatible with I^*. Then if for each equivalence class $a \in A^*$ we choose an arbitrary simple order S_a on the points of a, we obtain a simple order S on A compatible with I lexicographically by

$$xSy \leftrightarrow ([x] \neq [y] \,\&\, [x]\, R^*[y]) \vee ([x] = [y] = a \,\&\, xS_a y). \tag{5}$$

In order to characterize graphs compatible with a simple order, we recall here a definition used in our characterization of finite graphs representable in the form (3). Let us first say as in [6] that a point a in a graph (A, I) is an *extreme point* if, whenever x and y are adjacent to but not equivalent to a, then x is adjacent to y and there is some point adjacent to both x and y but not a. (Extreme points will correspond to points maximal or minimal in compatible simple orders.) Extending the definition given in [6] for finite graphs, let us say that a graph (A, I) is an *indifference graph* if whenever H is a finite, connected subgraph,[4] then either H^* has just one point or H^* has precisely two extreme points.

A main result of [6] is that for finite graphs the class of indifference graphs and the class of graphs representable in the form (3) are the same. It follows from the earlier discussion that every finite indifference graph is compatible with a simple order. We shall prove that the converse is also true and that the equivalence between the class of indifference graphs and the class of graphs compatible with some simple order extends to graphs of arbitrary cardinalities.

THEOREM 1. *A graph is compatible with a simple order if and only if it is an indifference graph.*

Proof. The proof that infinite indifference graphs are compatible with simple orders is most easily obtained from the result for finite indifference graphs by borrowing from the theory of models in formal logic. If $\mathscr{K}$ denotes the collection of all graphs compatible with some simple order, then it is possible to show (cf. the argument in Tarski [8, p. 586]) that $\mathscr{K}$ is axiomatizable by a set of universal sentences. But then by [8, Theorem 1.2], if all finite subgraphs of a graph G are in $\mathscr{K}$, we may conclude that G is in $\mathscr{K}$.[5]

To prove the converse, we note that it is sufficient to prove that every finite, connected, reduced graph with more than one point which is

[4] Subgraph will always mean "induced" subgraph, i.e., all adjacent lines (edges) are included.

[5] I am indebted to M. Jean for pointing out the above argument.

compatible with a simple order has precisely two extreme points. For a compatible simple order on a graph induces a compatible simple order on the reduction of each subgraph. Now if R is a compatible simple order on the finite, connected, reduced graph G, the points of G may be listed without repetitions as $c_1, c_2, \ldots, c_n$ so that $c_1 R c_2 R \cdots R c_n$. One now verifies that c_1 and c_n are extreme points, while $c_2, c_3, \ldots, c_{n-1}$ are not. Q.E.D.

COROLLARY 1.1. *Suppose $G = (A, I)$ is a finite graph and R is a simple order on A compatible with I. Then the points maximal or minimal for R are extremal in G. If G is reduced and connected, then the extreme points of G are maximal or minimal for R.*

Proof. This follows by the proof of the theorem, by passing if necessary to connected components in the reduction G^* of G. Q.E.D.

Remark. Goodman's [2] ideas suggest a characterization of the graphs compatible with simple orders, i.e., the indifference graphs, in terms of besideness rather than extremality. It is possible to prove, utilizing the Goodman definition of besideness, that a graph G is an indifference graph if and only if, whenever H is a finite, connected subgraph, then either (a) H^* has just one point or (b) each point of H^* is beside (relative to H^*) at least one and at most two others; and there are exactly two points of H^* which are beside only one other point.

3. THE UNIQUENESS THEOREM FOR COMPATIBILITY WITH A SIMPLE ORDER

We turn now to questions of uniqueness. Our aim is to prove that every connected indifference graph is compatible with essentially only one simple order.

LEMMA 1. *Suppose (A, I) is a finite, connected, reduced graph. Suppose R and R' are two simple orders on A compatible with I such that R and R' have the same maximal (minimal) element. Then $R = R'$.*

Proof. The points of A may be listed in the orders R and R' as $c_1, c_2, \ldots, c_n$ and $c_1', c_2', \ldots, c_n'$, respectively. By assumption $c_1 = c_1'$. Arguing by induction, suppose we have shown that $c_i = c_i'$, all $i < k$. We show $c_k = c_k'$. Since (A, I) is reduced, it is sufficient to show $c_k E c_k'$. Note that $c_k R c_k'$ and $c_k' R' c_k$, by inductive assumption. Let c_j be given.

We show $c_kIc_j \leftrightarrow c_k'Ic_j$. By symmetry, it is of course sufficient to show $c_kIc_j \rightarrow c_k'Ic_j$.

Note first that, by Corollary 1.1, c_{k-1} is an extreme point of the subgraph H generated by $\{c_i : i \geqslant k-1\}$. Note also that $c_{k-1}Ic_k$ and $c'_{k-1}Ic_k'$ by connectedness. By inductive assumption, $c_{k-1} = c'_{k-1}$ and c_k' is in H. Extremality of c_{k-1} in H then implies that c_kIc_k'.

To show that c_kIc_j implies $c_k'Ic_j$, suppose first that $j < k$. Then note that $c_jR'c_k'R'c_k$, so c_jIc_k' follows by compatibility. Next, if $j \geqslant k$, then we have either $c_k'Rc_j$ or c_jRc_k'. In either case, the result follows by compatibility. For in the former, we have $c_kRc_k'Rc_j$ and c_kIc_j, while in the latter we have $c_kRc_jRc_k'$ and c_kIc_k'. Q.E.D.

The *converse* of a simple order R on A is the simple order $\breve{R}$ on A defined by $\breve{R} = \{(x, y) \in A \times A: (y, x) \in R\}$.

THEOREM 2 (Uniqueness Theorem for Compatibility with a Simple Order). *Suppose $G = (A, I)$ is a connected, reduced indifference graph. If A has more than one point, then there are exactly two simple orders on A compatible with I, one the converse of the other.*

Proof. Fix $a \neq b$ in A and let R and R' be simple orders on A compatible with I and such that aRb and $aR'b$. There is at least one such simple order by Theorem 1 and the observation that, if a simple order is compatible with I, so is its converse. The theorem follows if we can show $R = R'$. Suppose first that A is finite and let c and c' be maximal for R and R', respectively. By the previous lemma, it is sufficient to prove that $c = c'$. Applying Corollary 1.1 in two directions, we see that c is either maximal or minimal for R'. The latter is impossible. For, if c is minimal for R', note that the simple order $S = \breve{R'}$ is compatible with I and that c is maximal for both R and S. Thus, by Lemma 1, $S = R$, and in particular aSb, whence $bR'a$. But then, since $aR'b$ also holds, we have $a = b$, contrary to assumption. Thus, c is maximal for R' and, since R' is simple, we conclude $c = c'$.

Suppose now G is infinite and let x, y be in A. It is easy to construct a finite, connected, reduced subgraph H of G containing a, b, x, y. The restrictions S and S' of R and R', respectively, to H are simple orders compatible with H. It follows by the finite case that $S = S'$ and so $xRy \leftrightarrow xR'y$. Thus, $R = R'$.

Remark. Theorem 2 is actually as strong a result as possible. In a compatible simple order we may interchange the order of two equivalent points or of two connected components without affecting compatibility.

4. The Oriented Case

It seems natural to study the compatibility between oriented graphs and simple orders as well as that between (unoriented) graphs and simple orders. This we turn to in the present section, obtaining results analogous to those above.

A pair (A, P) is an *oriented graph* if A is a set and P is an asymmetric binary relation on A. Its *symmetric complement* is the graph (A, I) defined by $I = \sim(P \cup \breve{P})$, where $\sim$ denotes set-theoretical complement and $\smile$ denotes converse. In particular, suppose A is a subset of the real numbers and P is defined on A by $xPy \leftrightarrow x > y + 1$. Then the symmetric complement is given by $xIy \leftrightarrow |x - y| \leqslant 1$. If R is the simple order "$\leqslant$", then we note that R is compatible with P in the sense that it is compatible with the symmetric complement I and moreover $xPy \rightarrow xRy$. This suggests the following definition, again stated for convenience in terms of the more general concept of weak order. Suppose (A, P) is an oriented graph and (A, I) is its symmetric complement. Suppose R is a weak order on A. Then we say that R is *compatible* with P if R is compatible with I and for all $x, y \in A$, xPy implies xRy. The oriented graphs H_1, H_2, H_3 of Figure 1 have the compatible simple orders 1, 2, 3, 4, 5 etc. shown and the respective symmetric complements G_1, G_2, G_3. As before, it is simple to prove that, if (A, P) is compatible with a weak order, then it is compatible with a simple order.

In [7], Scott and Suppes use the notion of *semiorder* introduced in Luce [5] and prove that the finite oriented graph (A, P) is a semiorder if and only if there is a real-valued function f on A so that for all $x, y \in A$, $xPy \leftrightarrow f(x) > f(y) + 1$. This suggests, as in the indifference graph case, that the semiorders will correspond to the oriented graphs compatible with some simple order.

Theorem 3. *Suppose (A, P) is an oriented graph. Then (A, P) is compatible with a simple order if and only if (A, P) is a semiorder.*

Proof. Let (A, I) be the symmetric complement. It is easy to see from the results of [6] that (A, P) is a semiorder if and only if (A, P) is transitive and (A, I) is an indifference graph. Suppose now (A, P) is compatible with a simple order R. Then (A, I) is compatible with R and so is an indifference graph. The transitivity of P is not hard to show using compatibility and we conclude (A, P) is a semiorder.

Conversely, suppose (A, P) is a semiorder. Scott and Suppes [7] define from (A, P) a natural weak order R on A by $xRy \leftrightarrow (\forall z)(zPx \rightarrow zPy \;\&\; yPz \rightarrow xPz)$. It turns out that R is actually compatible with P, and we leave it to the reader to check the details. Thus, P is compatible with a weak order and so, by an earlier observation, it is compatible with a simple order. Q.E.D.

Remark. Holland [4] proves the "if" part, using considerably different methods.

THEOREM 4 (Uniqueness Theorem for Compatibility of an Oriented Graph with a Simple Order). *Suppose (A, P) is a semiorder and its symmetric complement (A, I) is reduced. Then P is compatible with exactly one simple order.*

Proof. Let R and R' be two simple orders on A compatible with P. Suppose first that (A, I) is connected. Note that, if (A, I) is complete, then the result is trivial because in a reduced complete graph there is just one point. If (A, I) is not complete, there are a, b in A so that $\sim aIb$. Suppose without loss of generality that aPb. By compatibility, aRb and $aR'b$. Since in particular R and R' are compatible with I, and $R' \neq \breve{R}$, it follows by Theorem 2 that $R' = R$.

Suppose next that (A, I) is not connected. To show $R = R'$, let $x \neq y \in A$. We show $xRy \leftrightarrow xR'y$. This follows from the above proof if x and y are in the same component. If x and y are in different components, then $\sim xIy$. If xPy, then xRy and $xR'y$, whence $xRy \leftrightarrow xR'y$. If yPx, then yRx and $yR'x$. Also, $\sim xRy$ and $\sim xR'y$, since $x \neq y$. Thus, $xRy \leftrightarrow xR'y$. Q.E.D.

5. AN APPLICATION: REPRESENTING A GRAPH BY INTERVALS OF "JUST NOTICEABLE DIFFERENCE"

In this section we note how some of our results can be used to solve a problem suggested by Luce [5] and motivated by the notion of threshold or just noticeable difference (JND) in psychology. Suppose we interpret the points of a graph (A, I) as some set of "objects" we are comparing and the adjacency relation I as the relation "indifferent between." Luce's idea is to try to assign to each point x in A a value (real number) $f(x)$ and a threshold or JND (interval on the real line) about $f(x)$ so that we are indifferent between x and y if and only if $f(x)$ is within the threshold $J(y)$ of y. In this section we show when this can be done, and then give a similar analysis for the oriented analog.

Formally then, let us say that a graph (A, I) is *representable by JND's* if for each x in A there is a real number $f(x)$ and a (finite) real interval[6] $J(x)$ so that for all x, y in A, $xIy \leftrightarrow f(y) \in J(x)$. Note that, by the symmetry of I, $f(y) \in J(x)$ implies $f(x) \in J(y)$. This implication does not hold for arbitrarily chosen intervals on the real line with distinguished points inside.

The results of [6] show that every finite indifference graph is representable by JND's. For, if f satisfies Eq. (3), take $J(x) = [f(x) - 1, f(x) + 1]$. Here, the JND intervals are of uniform length and the value is in each case the midpoint. It will follow from Corollary 5.1 below that, in the finite case, if a JND representation can be obtained, then it can be obtained in such a uniform way.

LEMMA 2. *Suppose (A, I) is a graph. Then the following are equivalent:*

(a) *(A, I) is representable by JND's.*

(b) *There is a weak order R on A compatible with I and a real-valued function f on A so that, for all $x, y \in A$, $xRy \leftrightarrow f(x) \geqslant f(y)$.*

Proof. To show that (a) implies (b), suppose (A, I) is representable by f, J. *Define* a weak order R on A by $xRy \leftrightarrow f(x) \geqslant f(y)$. The verification that R is compatible with I is not difficult.

To prove that (b) implies (a), note first that without loss of generality we may take the range of f to be a subset of the interval $(0, 1)$ simply by mapping the entire real line into $(0, 1)$ in an order-preserving fashion. Then f is bounded, and we may, following Luce [5], define $\bar{\delta}(x) = \sup\{f(y) : xIy\} - f(x)$ and $\underline{\delta}(x) = f(x) - \inf\{f(y) : xIy\}$. One can verify that (A, I) is representable by f and $J(x) = \{f(x) - \underline{\delta}(x), f(x) + \bar{\delta}(x)\}$, with appropriate choice of closed or open on each end. Closedness or openness may be decided according to the following rule: $f(x) - \underline{\delta}(x)$ is in $J(x)$ if and only if $f(x) - \underline{\delta}(x) = f(y)$ for some y adjacent to x; and similarly for $f(x) + \bar{\delta}(x)$. Q.E.D.

The lemma leads directly to a criterion for JND representability for connected graphs. Let $G = (A, I)$ be a connected indifference graph. Then $G^* = (A^*, I^*)$ is also a connected indifference graph, and so by Theorem 2 it is compatible with exactly two simple orders, one the converse of the other. (There is only one if A has just one point.) Denote by $S(G)$ either of these simple orders. We have

THEOREM 5 (Criterion for Representability by JND's). *Suppose G is a connected graph. Then G is representable by JND's if and only if G is an*

[6] We make no restriction on the boundary of the interval $J(x)$, and indeed allow open, closed, or half-open intervals. It is always possible to modify a JND representation so that the intervals are all open (all closed) provided the graph is countable, but not in general otherwise.

indifference graph and $S(G)$ is order-isomorphic to a subset of the real numbers.

Remark. A nice criterion, due to Birkhoff [1, p. 23], exists for determining when a given simple order is order-isomorphic to a subset of the real numbers.

COROLLARY 5.1. *Suppose G is a countable graph. Then G is representable by JND's if and only if it is an indifference graph.*

Proof. Use the standard result (cf. Birkhoff [1]) that every countable simple order is order-isomorphic to a subset of the real numbers. Q.E.D.

The above results have analogs in the oriented case. If (A, P) is an oriented graph and (A, I) is its symmetric complement, let us say, again motivated by Luce [5], that (A, P) is *representable by JND's* if for each x in A there is a real number $f(x)$ and a (finite) real interval $J(x)$ so that (f, J) represents (A, I) by JND's and so that for all x, y in A, $xPy \leftrightarrow f(x) > J(y)$, where $f(x) > J(y)$ means $f(x) > a$ for all $a \in J(y)$. The proofs here are virtually the same as for the corresponding unoriented results, or are simple deductions from these results.

LEMMA 3. *Suppose (A, P) is an oriented graph. Then the following are equivalent:*

(a) *(A, P) is representable by JND's.*

(b) *There is a weak order R on A compatible with P and a real-valued function f on A so that for all $x, y \in A$, $xRy \leftrightarrow f(x) \geqslant f(y)$.*

This result once again leads to a criterion for representability by JND's. Let $H = (A, P)$ be a semiorder. If P^* is defined on A^* by $[x]\, P^*[y] \leftrightarrow xPy$, then P^* is also a semiorder and I^* is its symmetric complement. It follows by Theorem 4 that there is a unique simple order on A^* compatible with P^*. We denote this simple order by $T(H)$. Then we have

THEOREM 6 (Criterion for JND Representability of Oriented Graphs). *Suppose $H = (A, P)$ is an oriented graph. Then H is representable by JND's if and only if H is a semiorder and $T(H)$ is order-isomorphic to a subset of the real numbers.*

COROLLARY 6.1. *Suppose H is a countable oriented graph. Then H is representable by JND's if and only if it is a semiorder.*

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